Section 1.7 Linear Independence

Definition of Linearly Independence

An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Example 1. Find the value of h for which the vectors are linearly dependent. Justify each answer.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}$$

ANS: By the definition of linearly dependent. we need to find h such that.

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

has a nontrivial solution.

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & \cancel{2}\ 1 & \cancel{2}\ 1 & 0 \\ 0 & 23 & h-3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{array}\right]$$

Then the equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$

has a nontrivial solution if and only if $h-26=0$ (which means x_3 is a free varible).

Thus $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent if and only if $h=26$.

Rmk: If the question ask us to find the value of h such that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. this corresponds to finding h such that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

has only trivial solution.

That means we have no free variable in the system. i.e. $h-26 \neq 0$.

Linear Independence of Matrix Columns

Suppose a matrix $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following:

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example 2. Determine if the columns of the matrix form a linearly independent set. Justify your answer.

$$A = \begin{array}{ccc} \multicolumn{3}{c}{\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}} \\ \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array}$$

ANS: By the above discussion, we know

The columns of A are linearly independent

$\iff A\vec{x} = \vec{0}$ has only trivial solution.

The augmented matrix for $A\vec{x} = \vec{0}$ is

$$\left[\begin{array}{ccc|c} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Note: The corresponding system has 3 variables and 4 equations.

The correspond system is

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases}$$

There are no free varibles. So $A\vec{x}=\vec{0}$ only has trivial solution.

Thus the columns of A are linearly independent.

$\vec{v}_1\ \vec{v}_2$ linearly dep $\iff$ $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for some $c_1\ c_2$ not all 0s.
Say $c_1 \neq 0$. Then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2$

Sets of One or Two Vectors

1. A set containing only one vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v}$ is not the zero vector.
2. A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sets of Two or More Vectors

For ex, $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ with $c_1 \neq 0$, then $\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \frac{c_3}{c_1}\vec{v}_3$

Theorem 7 Characterization of Linearly Dependent Sets
An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Theorem 8. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 9. If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

If $\vec{v}_1 = \vec{0}$, then $c_1 = 1$, $c_2 = c_3 = 0$ is a nonzero sol to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

Example 3. Determine by inspection whether the vectors are linearly independent. Justify each answer.

a. $\underbrace{\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}}_{\vec{v}_2}$ (set of two vectors)

(a) Note $\frac{4}{6} = \frac{-2}{-3} = \frac{6}{9} = \frac{2}{3}$.
So $\vec{v}_1 = \frac{2}{3}\vec{v}_2$
They are linearly dependent.

b. $\begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

By Thm 8, the given vectors are linearly dependent. since there are 4 vectors but only 2 entries in each vector.

c. $\begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By Thm 9, the given vectors are linearly dependent since the list contains a zero vector.

Example 4. Given $A = \begin{array}{c} \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \\ \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$, observe that the third column is the sum of the first two columns.

Find a nontrivial solution of $A\mathrm{x} = 0$.

ANS: Let $\vec{a}_1, \vec{a}_2, \vec{a}_3$ be the columns of A, respectively.

Then $A\vec{x} = \vec{0} \iff x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{0}$, where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

We know $\vec{a}_3 = \vec{a}_1 + \vec{a}_2 \iff \vec{a}_1 + \vec{a}_2 - \vec{a}_3 = \vec{0}$.

So we can take

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

as a nontrivial solution to $A\vec{x} = \vec{0}$.

The following two questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5. Describe the possible echelon forms of the matrix. Use the notation of Example 1 in Section 1.2.

a. A is a 3×3 matrix with linearly independent columns.

b. A is a 4×2 matrix, $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$, and $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

Solution: a) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

b) $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & * \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Exercise 6.

1. How many pivot columns must a 6×4 matrix have if its columns are linearly independent? Why?

Solution. All 4 columns of the 6×4 matrix A must be pivot columns. Otherwise, the equation $A\vec{x} = \vec{0}$ would have a free varible, in which case the columns of A would be linearly dependent.

2. How many pivot columns must a 4×6 matrix have if its columns span $\mathbb{R}^4$? Why?

Solution. If the colums of a 4×6 matrix A span $\mathbb{R}^4$, then A has a pivot in each row, by Theorem 4. Since each pivot position is in a different column, A has 4 pivot column.